INTEGRALLASH USULLARI

Saidov Yuldash Razzaqovich
fizika-matematika fanlari nomzodi, dotsent
Xorazm viloyati PYMO'MM direktori
Telefon: +998 (97)458-89-57
E-mail: yuldashyuldash6080@gmail.com

Saidov Sherzodbek Yuldash O'g'li
Urganch Davlat Universiteti talabasi
Telefon: +998 (94)230-57-07
E-mail: saidovsherzod936@gmail.com

Annotatsiya: Mazkur maqolada integrallash va ularga doir misollar yechish usullari haqida so'z yuritiladi.

Kalit so'zlar: diferensiallanuvchi funksiya, integral, bo'laklab integrallash formulasi, ratsional kasrlarni integrallash, integrallar jadvali, yangi o'zgaruvchi, o'rniga qo'yish

Integrallash ham matematikaning asosiy tushunchalaridan bo'lib, matematika va fizikada egri chiziq uzunligi va qattiq jismning hajmini topish kabi masalalarda qo'llaniladi. Integrallashda quyidagi usullardan ko'proq foydalaniladi:
1) Bevosita integrallash;
2) Bo'laklab integrallsh;
3) O'rniga qo'yish(belgilash);
4)Trigonometrik integrallarni topish;
5)Ratsional kasrlarni integrallsh…

Ularning har biriga atroflicha to'xtalib o'tamiz…

Bevosita integrallash usuli

Bevosita integrallash usulida integral ostidagi funksiyani formulalar yordamida almashtirish hamda aniqmas integralning asosiy xossalari va integrallar jadvalidan foydalanib integral topiladi.

Quyida bir misolning yechilishi billan tanishamiz.

$$\int \frac{dx}{\sin^2 x\cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x\cos^2 x}dx = \int\left(\frac{\sin^2 x}{\sin^2 x\cos^2 x} + \frac{\cos^2 x}{\sin^2 x\cos^2 x}\right)dx =$$

$$\int\left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right)dx = \int\frac{dx}{\cos^2 x} + \int\frac{dx}{\sin^2 x} = tgx - ctgx + c.$$

Bo‘laklab integrallsh usuli.

Quyida e’tiboringizga bo‘laklab integrallash usuli haqida ma’lumot berib o‘tamiz.

Bizga ikkita diferensiallanuvchi u(x) va v(x) funksiyalar berilgan .bo‘lsin. Bu funksiyalar ko‘paytmasi (uv) ning differensialini topaylik.

d(*uv*)=*udv*+*vdu*

Buni ikki tomonini hadma-had integrallab quyidagini topamiz:

$$uv = \int udv + \int vdu \quad \text{yoki} \quad \int udv = uv - \int vdu \qquad (1)$$

Oxirgi topilgan ifoda bo‘laklab integrallash formulasi deyiladi.

Bu formulani qo‘llab integral hisoblaganda $\int udv$ yoki $\int vdu$ ko‘rinishdagi integralga keltiriladi.

Integral ostida *u*=ln*x* funksiya, yoki ikkita funksiyaning ko‘paytmasi, hamda teskari trigonometrik funksiyalar qatnashgan bo‘lsa,bunda bo‘laklab integrallash formulasi qo‘llaniladi. Aniqmas integralni hisoblaganda topilgan natija yoniga o‘zgarmas (*C*=const) ni qo‘shib qo‘yish shart. Aks holda integralning bitta qiymati topilib, qolganlari tashlab yuborilgan bo‘ladi. Bu esa integrallashda xatolikka yo‘l qo‘yilgan deb hisoblanadi.

Quyidagi masalani ko‘rib chiqamiz:

$\int xarctgxdx$ ni hisoblang. Bo‘laklab integrallash usuliga asosan,

$$u = arctgx \quad dv = xdx \quad du = \frac{dx}{1+x^2} \quad v = \int xdx = x^2/2$$

(bunda *C*=0 deb olindi)

(1) formulani qo‘llaymiz

$$\int xarctgxdx = \frac{x^2}{2} arctgx - \int \frac{x^2}{2(1+x^2)} dx \qquad (*)$$

$$\int \frac{x^2}{1+x^2} dx$$

ni alohida hisoblaymiz

$$\int \frac{x^2}{1+x^2} dx = \int \frac{1+x^2-1}{1+x^2} dx = \int (1 - \frac{1}{1+x^2}) dx = x - arctgx + C$$

buni (*)ga qo‘yamiz.

$$\int xarctgxdx = \frac{x^2}{2}arctgx - \frac{x}{2} + \frac{1}{2}arctgx + C = -\frac{x}{2} + \frac{x^2+1}{2}arctgx + C$$

$\int(3x^5 + 4x - 3)dx = \int 3x^5dx + \int 4x\,dx$ - $\int 3\,dx$ =3$\int x^5dx$ +4$\int xdx - 3\int dx$=3 $\frac{x^6}{6}$ +$\frac{4x^2}{2}$ -3x+C=$\frac{1}{2}x^6$ +2x^2 -3x+C

O‘rniga qo‘yish (belgilab olish)

Bu metod asosiy usul sanaladi.Agar integralni bevosita topib bo‘lmasa, Ya’ni u jadvaliy bo‘lmasa o‘rniga qo‘yish usuliga murojaat qilinadi. I=$\int L(x)$dx integralni topish talab qilinayotgan bo‘lsin.x=$\varphi(t)$ kabi belgilash kiritamiz. $\varphi(t)$ differensiallanuvchi funksiya.Quyidagi qoidaga ega bo‘lamiz:

Agar $\int \boldsymbol{L(x)dx}$=F(x)+C (a) bo‘lsa, $\int f[\varphi(t)]\varphi'(t)$dt= F($\boldsymbol{\varphi(t)}$)+C (1) bo‘ladi. d[$\boldsymbol{F(x)}$]=L(x)dx. U holda differensiallsh formulasining invariantligiga ko‘ra: d{$\boldsymbol{F[\varphi(t)]}$}=L[$\boldsymbol{\varphi(t)}$]d$\boldsymbol{\varphi(t)}$=L[$\boldsymbol{\varphi(t)}$]***φ’(t)*dt** Bu tenglikdan (1) ning to‘g‘riligi kelib chiqadi. Bu teorema quyidagi ikkita o‘rniga quyish usuliga asos bo‘ladi:x=φ(t) belgilash, U holda $\int f(x)$dx=$\int f[\varphi(t)]\cdot\varphi'(t)$dt. Bu integral topilgach,o‘ng tomondagi t o‘rniga eski x o‘zgaruvchining qiymati qo‘yiladi.

2)φ(x)=t deb belgilash.U holda $\int f[\varphi(x)]\cdot\varphi'(x) = \int f(t)$dt. Bu integral topilgach,t o‘rniga $\varphi(x)$ qo‘yiladi.

$$\int(\ln x)^4\frac{dx}{x}\left|\begin{matrix}\ln x = t, d(\ln x) = dt,\\ (\ln x)'dx = dt, \frac{1}{x}dx = dt\end{matrix}\right| = \int t^4dt = \frac{t^5}{5} + c = \frac{\ln^5 x}{x} + c$$

Trigonometrik funksiyalarning integrallari

Trigonometrik funktsiyalarning ko'paytmasini ularning yig'indisiga aylantirish uchun formulalardan foydalanamiz:

sin a cos b = ½(sin(a+b)+sin(a-b))

cos a cos b = ½(cos(a+b)+cos(a-b))

sin a sin b = ½(cos(a-b)-cos(a+b))

∫ cos 4 x sin 3 xdx integralini hisoblang. Cos(x)=t almashtirishni qilamiz. U holda ∫ cos 4 x sin 3 xdx

$$= -\int t^4(1-t^2)dt = \frac{t^7}{7} - \frac{t^5}{5} + C = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C.$$

2. Integralni hisoblang.
Sin x=t almashtirishni hosil qilib, olamiz

$$= -\frac{1}{3\sin^3 x} + \frac{1}{\sin x} + C.$$

Ratsional kasrlarni integrallash

Surat va maxraji:

P(x)=$\boldsymbol{a_n x^n}$+$\boldsymbol{a_{n-1} x^{n-1}}$+…….+$\boldsymbol{a_1}$x+$\boldsymbol{a_0}$

Q(x)=$\boldsymbol{b_m x^m}$+$\boldsymbol{b_{m-1} x^{m-1}}$+…….+bx+$\boldsymbol{b_0}$

Ko‘phadlardan iborat $\frac{P(x)}{Q(x)}$ kasrga ratsional kasr deyiladi. Agar n<m bo‘lsa, bu oddiy kasr bo‘ladi.n≥m bo‘lganda esa, u noto‘g‘ri kasr bo‘ladi.

Ikkita ko‘phadning nisbati ratsional kasr deyiladi. $\frac{P_n(x)}{Q_m(x)}$. Bunda $n<m$ bo‘lsa ratsional kasr to‘g‘ri ratsional kasr deyiladi, $n \geq m$ bo‘lsa, ratsional kasr noto‘g‘ri ratsional kasr deyiladi. Bunday kasr suratini maxrajiga bo‘lish bilan butun va kasr qismlarga ajratiladi. Bundagi kasr to‘g‘ri kasr bo‘ladi. Agar ratsional kasrda maxraj ya’ni $Q_m(x)$ =1 bo‘lsa, kasr butun ratsional funksiyaga aylanadi. Buni integrallash yuqorida ko‘rib o‘tilgan.

Endi to‘g‘ri ratsional kasrni integrallashni ko‘rib o‘tamiz. Avval oddiy ratsional kasrlarni integrallashni ko‘ramiz. Umumiy holda ratsional kasr oddiy kasrlarga ajratilib, so‘ngra integrallanadi. Oddiy ratsional kasrlar (ba’zan elementar kasrlar deb ham yuritiladi) qo‘yidagi ko‘rinishda bo‘ladi.

1. $\frac{A}{x-a}$ 2. $\frac{A}{(x-a)^k}$ bu yerda $k \geq 2$ butun musbat son.

3. $\frac{Ax+B}{x^2+px+q}$. Maxrajni ildizi kompleks sonlardan iborat, ya’ni $\frac{p^2}{4} - q \leq 0$

4. $\frac{Ax+B}{(x^2+px+q)^k}$. $k \geq 2$ bo‘lgan butun musbat son.

(1)-(4) ko‘rinishdagi kasrlar eng sodda ratsional kasrlardir.

n≥m bo‘lganda P(x) ni Q(x) ga bo‘lib, uni hamma vaqt oddiy kasrga keltirish mumkin.Ana shunday kasrlar qatnashgan aniqmas integrallarning ayrim muhim xususiy hollarini qarab chiqamiz.

Masalan, $\int \frac{4x^3-19}{x^3(x-5)}dx.$ **Bu holda** $\frac{4x^3-19}{x^3(x-5)} = \frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{D}{x-5} = \frac{A(x-5)+Bx(x-5)+Cx^2(x-5)+Dx^3}{x^3(x-5)} = \frac{(C+D)x^3+(B-5C)x^2+(A-5B)x-5A}{x^3(x-5)}$. **A=19/5;B=1/5;C=19/125;D=491/125 shunday qilib,** $\int \frac{4x^3-19}{x^3(x-5)}dx = \int (\frac{19}{5}\cdot\frac{1}{x^3} + \frac{19}{25}\cdot\frac{1}{x^2} + \frac{19}{125}\cdot\frac{1}{x} + \frac{481}{125}\cdot\frac{1}{x-5})dx.$

C+D=4

B-5C=0

A-5B=0

5A=19

Endi shu kasrlarni integrallashni ko‘raylik.

$$1.\int \frac{A}{x-a}dx = A\ln|x-a| + C \qquad (C = const)$$

$$2.\int \frac{A}{(x-a)^k}dx = A\int (x-a)^{-k}d(x-a) =$$

$$= A\frac{(x-a)^{-k+1}}{-k+1} + C = \frac{A}{(1-k)(x-a)^{k-1}} + C$$

$$3.\int \frac{Ax+B}{x^2+px+q}dx = \int \frac{\frac{A}{2}(2x+p) + (B - \frac{Ap}{2})}{x^2+px+q}dx = \frac{A}{2}\int \frac{2x+p}{x^2+px+q}dx +$$

$$+ (B - \frac{Ap}{2})\int \frac{dx}{x^2+px+q} = \left[\begin{matrix} x^2+px+q = t \\ (2x+p)dx = dt \end{matrix} \quad \int \frac{dt}{t} = \ln|t| + C\right] =$$

$$= \frac{A}{2}\ln|x^2+px+q| + (B - \frac{Ap}{2})\int \frac{dx}{(x+p/2)^2 + (q-p^2/4)} =$$

$$= \frac{A}{2}\ln|x^2+px+q| + \frac{2B-Ap}{\sqrt{4q-p^2}}arctg\frac{2x+p}{\sqrt{4q-p^2}} + C$$

$$4.\int \frac{Ax+B}{(x^2+px+q)^k}dx = \int \frac{A/2(2x+p)+B-Ap/2)}{(x^2+px+q)^k}dx =$$

$$= \frac{A}{2}\int \frac{2x+p}{(x^2+px+q)^k}dx + (B-\frac{Ap}{2})\int \frac{dx}{(x^2+px+q)^k}.$$

$\int \frac{3x^2+8}{x^3+4x^2+4x}dx$ hisoblansin.

Yechish. Integral ostidagi kasrning maxrajini ko‘paytuvchilarga ajratamiz. $x^3+4x^2+4x=x(x^2+4x+4)=x(x+2)^2$

Endi berilgan kasrni (*) dan foydalanib, elementar kasrlarga yoyamiz:

$$\frac{3x^2+8}{x(x+2)^2} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2} \qquad (**)$$

buni har ikki tomonini $x(x+2)^2$ ga ko‘paytiramiz.

$3x^2+8=A(x+2)^2+Bx(x+2)+Cx=(A+B)x^2+(4A+2B+C)x+4A$

Endi x ni bir xil darajalari oldidagi koeffitsientlarini tenglashtirib, tenglamalar sistemasini tuzamiz:

$$\begin{cases} A+B=3 \\ 4A+2B+C=0 \\ 4A=8 \end{cases}$$

Bu sistemani yechib, A=2, B=1, C=-10 larni topamiz. So‘ngra bularni (**) ga qo‘yamiz.

$$\frac{3x^2+8}{x(x+2)^2} = \frac{2}{x} + \frac{1}{x+2} - \frac{10}{(x+2)^2}$$

Buni ikki tomonini dx ga ko‘paytirib, keyin integrallaymiz:

$$\int \frac{3x^2+8}{x^3+4x^2+4x}dx = \int [\frac{2}{x} + \frac{1}{x+2} - \frac{10}{(x+2)^2}]dx =$$

$$= 2\int \frac{dx}{x} + \int \frac{dx}{x+2} - 10\int (x+2)^{-2}d(x+2) =$$

$=2\ln|x|+\ln|x+2|+10/(x+2)+C$

Misol 2. $\int \frac{x^3+4x^2-2x+1}{x^4+x}dx$ hisoblansin.

Yechish. Berilgan kasrni elementar kasrlarga ajratamiz. Buning uchun maxrajdagi ko‘phadni ko‘paytuvchilarga ajratamiz

$x^4+x=x(x^3+1)=x(x+1)(x^2-x+1)$

$$\frac{x^3+4x^2-2x+1}{x^4+x}=\frac{A}{x}+\frac{B}{x+1}+\frac{Cx+D}{x^2-x+1}$$

$x^3+4x^2-2x+1=A(x^3+1)+Bx(x^2-x+1)+(Cx+D)(x^2+x)=$

$=(A+B+C)x^3+(C+D-B)x^2+(B+D)x+A$

Endi x larning bir xil darajalari oldidagi koeffitsientlarni tenglashtirish bilan noma’lum A, B, C, D larni aniqlash uchun qo‘yidagi 4 ta tenglamani hosil qilamiz, hamda bu tenglamalar sistemasini yechib, A, B, C, D noma’lumlarni aniqlaymiz:

$$\begin{cases} A+B+C=1 \\ C+D-B=4 \\ B+D=-2 \\ A=1 \end{cases} \qquad \begin{matrix} A=1 \\ B=-2 \\ C=2 \\ D=0 \end{matrix}$$

Topilganlarni noma’lumlar o‘rniga qo‘yib kasrni elementar kasrlar orqali ifodasini yozamiz:

$$\frac{x^3+4x^2-2x+1}{x^4+x}=\frac{1}{x}-\frac{2}{x+1}+\frac{2x}{x^2-x+1}$$

endi buni integrallaymiz.

$$J=\int \frac{x^3+4x^2-2x+1}{x^4+x}dx=\int \frac{dx}{x}-2\int \frac{dx}{x+1}+2\int \frac{xdx}{x^2-x+1}=$$

$=\ln|x|-2\ln|x+1|+2J_1$

J_1 integralda x^2-x+1 dan to‘la kvadrat ajratamiz:

$x^2-x+1=(x-1/2)^2+3/4$ Bunda $x-1/2=t$ $dx=dt$ deb olamiz. U holda J_1 qo‘yidagicha hisoblanadi.

$$J_1=\int \frac{(t+1/2)dt}{t^2+3/4}=\frac{1}{2}\int \frac{2tdt}{t^2+3/4}+\frac{1}{2}\int \frac{dt}{t^2+3/4}=\frac{1}{2}\ln(t^2+3/4)+$$
$$+\frac{1}{\sqrt{3}}arctg\frac{2t}{\sqrt{3}}=\frac{1}{2}\ln(x^2-x+1)+\frac{1}{\sqrt{3}}arctg\frac{2x-1}{\sqrt{3}}.$$

Natijada J integral qo‘yidagicha aniqlanadi:

$$J=\ln\frac{|x|(x^2-x+1)}{(x+1)^2}+\frac{2}{\sqrt{3}}arctg\frac{2x-1}{\sqrt{3}}+C$$

Miso l.3. $\int \frac{(x^3-3)dx}{x^4+10x^2+25}$ hisoblansin.

Yechish. Maxrajni ko‘paytuvchilarga ajratamiz:

$$x^4+10x^2+25=(x^2+5)^2$$

$$\frac{x^3-3}{x^4+10x^2+25}=\frac{Ax+B}{x^2+5}+\frac{Cx+D}{(x^2+5)^2}$$

$$x^3-3=(Ax+B)(x^2+5)+Cx+D=Ax^3+Bx^2+(5A+C)x+(5B+D)$$

$$\begin{cases} A=1 & A=1 \\ B=0 & B=0 \\ 5A+C=0 & C=-5 \\ 5B+D=-3 & D=-3 \end{cases}$$

$$\frac{x^3-3}{x^4+10x^2+25}=\frac{x}{x^2+5}-\frac{5x+3}{(x^2+5)^2}$$

Buni integrallaymiz:

$$J=\int\frac{x^3-3}{x^4+10x^2+25}dx=\underbrace{\int\frac{xdx}{x^2+5}}_{J_1}-5\underbrace{\int\frac{xdx}{(x^2+5)^2}}_{J_2}-3\underbrace{\int\frac{dx}{(x^2+5)^2}}_{J_3}.$$

$$J_1=\int\frac{xdx}{x^2+5}=\frac{1}{2}\int\frac{2xdx}{x^2+5}=\frac{1}{2}\int\frac{d(x^2+5)}{x^2+5}=\frac{1}{2}\ln(x^2+5)$$

$$J_2=\int\frac{xdx}{(x^2+5)^2}=\frac{1}{2}\int(x^2+5)^{-2}d(x^2+5)=\frac{1}{2}\frac{(x^2+5)^{-1}}{-1}=-\frac{1}{2(x^2+5)}$$

J_3 integralda o‘zgaruvchini almashtiramiz:

$$x=\sqrt{5}tgz\;dx=\sqrt{5}\sec^2 zdz$$

$$J_3=\int\frac{dx}{(x^2+5)^2}=\int\frac{\sqrt{5}\sec^2 zdz}{25\sec^4 z}=\frac{1}{5\sqrt{5}}\int\cos^2 zdz=$$

$$=\frac{1}{10\sqrt{5}}\int(1+\cos 2z)dz=\frac{1}{10\sqrt{5}}(z+\frac{1}{2}\sin 2z)=\frac{1}{10\sqrt{5}}(arctg\frac{x}{\sqrt{5}}+\frac{x\sqrt{5}}{x^2+5}).$$

Berilgan integral qo‘yidagiga teng bo‘ladi:

$$J=\frac{1}{2}\ln(x^2+5)+\frac{5}{2(x^2+5)}-\frac{3}{10\sqrt{5}}(arctg\frac{x}{\sqrt{5}}+\frac{x\sqrt{5}}{x^2+5})+C=$$

$$=\frac{1}{2}\ln(x^2+5)+\frac{25-3x}{10(x^2+5)}-\frac{3}{10\sqrt{5}}arctg\frac{x}{\sqrt{5}}+C.$$

Integrallash usullari haqida ko‘plab gapirish mumkin.Mazkur maqolada ularning ba’zilari haqida so‘z yuritdik...

Foydalanilgan adabiyotlar ro‘yxati:

1. Т.Азларов, Ҳ. Мансуров “Математик анализ”1-қисм. Тошкент,1986
2. А.А. Гусак. Высшая математика.1-том.Минск,2001
3. В.П.Минорский,Сборник задач по высшей математике.Москва. “Наука”,2000